Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

ON CONSENSUS IN THE CUCKER–SMALE TYPE MODEL ON ISOLATED TIME SCALES

Ewa Girejko

Faculty of Computer Science,
Bialystok University of Technology,
15-351 Białystok, Poland

Luís Machado

Department of Mathematics, UTAD,
5001-801 Vila Real, Portugal

Agnieszka B. Malinowska*

Faculty of Computer Science,
Bialystok University of Technology,
15-351 Białystok, Poland

Natália Martins

Center for Research and Development in Mathematics and Applications,
Department of Mathematics,
University of Aveiro,
3810-193 Aveiro, Portugal

(Communicated by the associate editor name)

Abstract. This article addresses a consensus phenomenon in a Cucker-Smale model where the magnitude of the step size is not necessarily a constant but it is a function of time. In the considered model the weights of mutual influences in the group of agents do not change. A sufficient condition under which the proposed model tends to a consensus is obtained. This condition strikingly demonstrates the importance of the graininess function in a consensus phenomenon. The results are illustrated by numerical simulations.

1. **Introduction.** Recently we can observe quite intensive increase of interest in research on groups of autonomous agents, in which everyone influences on others' behavior and opinions. The mechanism of cooperation in the group of agents seems to be very interesting to study: it creates qualitatively new behaviors like reaching a consensus without any central direction. Examples of this are the emergence of common languages in primitive societies, the way in which populations of animals move together (referred as flocking for birds) or reaching a consensus in a group of people like party or society. The description of this kind of interaction between agents is well suited by opinion dynamics. First steps towards the creation of opinion dynamics models were made in 1956 by John French (see [18]), who

2010 *Mathematics Subject Classification.* Primary: 39A12; Secondary: 34N05.

Key words and phrases. Consensus formation, opinion dynamics, Cucker–Smale model, time scale.

* Corresponding author: a.malinowska@pb.edu.pl.

studied the interpretation of many complex phenomena about groups taking into account interpersonal relations among the individuals. Since then many researchers have been studying the behavior of a group of autonomous agents and a consensus formation [14, 17, 26, 31]. In 1997 Krause [25] presented a nonlinear opinion dynamics model under bounded confidence, that is, each agent's opinion is influenced by those opinions, which are not too far. In the literature this model is also known as the Hegselmann–Krause model because of the profound study that was done by Hegselmann and Krause in [23]. In recent years several authors dedicated their attention to the study of Krause's model, see for example [6, 7, 19, 20, 30] and references therein. An important generalization of the Krause model was presented by Cucker and Smale in [15]. In this case the dynamics is described by a system of two equations:

$$\begin{cases} x_i(t+h) - x_i(t) = hv_i(t) \\ v_i(t+h) - v_i(t) = h\displaystyle\sum_{j=1}^{N} a_{ij}\big(v_j(t) - v_i(t)\big), \end{cases} \tag{1}$$

for $i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^k$ and $v_i(t) \in \mathbb{R}^k$ denote, respectively, the state and consensus parameters of agent i at time t, h is the magnitude of the step size and the weights $a_{ij} = \dfrac{H}{\big(1 + \|x_i - x_j\|^2\big)^{\beta}}$, for fixed $H > 0$ and $\beta \geq 0$, quantify the way that the members of the group influence each other. Cucker and Smale in [16] showed that under certain conditions the convergence of the consensus parameters to a common value is guaranteed.

In this paper we propose the following Cucker–Smale type model on an isolated time scale $\mathbb{T}$:

$$\begin{cases} x_i(\sigma(t)) = x_i(t) + \mu(t)v_i(t) \\ v_i(\sigma(t)) = v_i(t) + \frac{\mu(t)}{N}\displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j(t) - v_i(t)\right), \end{cases} \tag{2}$$

for $i = 1, 2, ..., N$, where σ and μ denote, respectively, the forward jump operator and the graininess function on $\mathbb{T}$ (for the definitions see Section 2), and $a_{ij} \in \mathbb{R}_0^+ := [0, \infty)$, for $i, j \in \{1, ..., N\}$. In other words, the magnitude of the step size is a function of time (not necessarily a constant). Our aim is to analyze how the convergence to a consensus depends on the graininess function, under the assumption that the way in which the members of the group influence each other does not change. As a motivating example, let us consider model (2) with $N = 5$ and the weights a_{ij} given by

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Matrix $\bar{A}$ describes a situation when agent 1 influences agents 2,3,4,5 and agent 2 influences agent 5 and no other influence occurs. Figures 1–2 illustrate the behavior of the system on two different time scales with the same initial conditions. The performed numerical simulations suggest that on $\mathbb{T} = \left\{1 - \frac{1}{k} : k = 1, \ldots, 10\right\} \cup$

$\left\{t_k = 1 + 2.5\sum_{i=0}^{k}\left|\sin(i)\right| \ : \ k \in \mathbb{N}_0\right\}$ the system tends to a consensus, whereas on $\mathbb{T} = \left\{1 - \frac{1}{k} \ : \ k = 1, \ldots, 10\right\} \cup \left\{t_k = 1 + 6\sum_{i=0}^{k}\left|\sin(i)\right| \ : \ k \in \mathbb{N}_0\right\}$ there is no consensus. This example shows strikingly how important is the choice of a time scale in a consensus phenomenon.

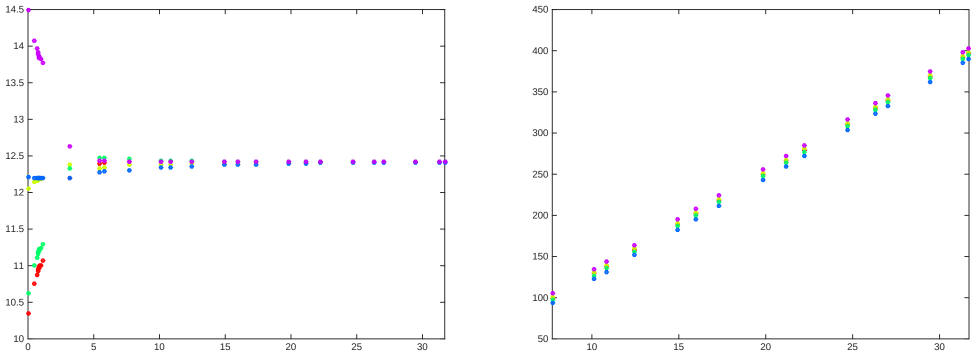

FIGURE 1. Time evolution of 5 consensus parameters with 30 iterations (left) and their states in the last 16 iterations (right) on $\mathbb{T} = \left\{1 - \frac{1}{k} \ : \ k = 1, \ldots, 10\right\} \cup \left\{t_k = 1 + 2.5\sum_{i=0}^{k}\left|\sin(i)\right| \ : \ k \in \mathbb{N}_0\right\}$.

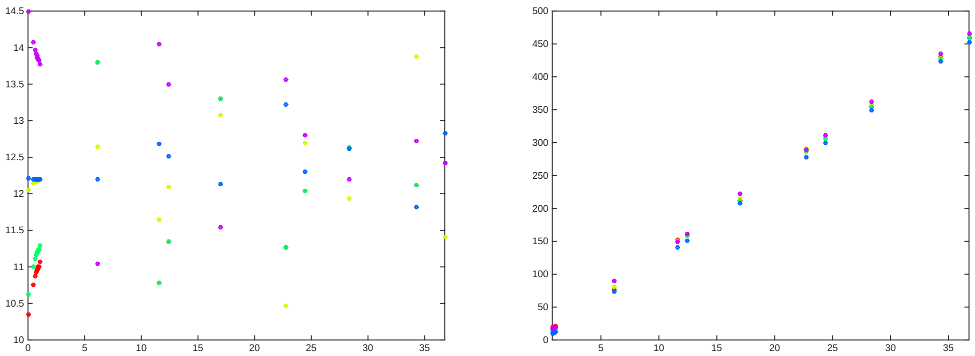

FIGURE 2. Time evolution of 5 consensus parameters with 20 iterations (left) and their states in the last 16 iterations (right) on $\mathbb{T} = \left\{1 - \frac{1}{k} \ : \ k = 1, \ldots, 10\right\} \cup \left\{t_k = 1 + 6\sum_{i=0}^{k}\left|\sin(i)\right| \ : \ k \in \mathbb{N}_0\right\}$.

Our discussions will be organized as follows: Section 2 sets up notation and terminology. In Section 3 we prove a sufficient condition (regarding the range of the graininess function) under which the considered model tends to a consensus (Theorem 3.2) and we apply this condition to specific cases of system (2). Moreover, it is proved that under certain conditions all consensus parameters converge to a weighted average mean, which is proved to be an invariant of the dynamics. Several examples showing the behavior of dynamic system (2) on different time scales are given in order to illustrate the results. We end with Section 4 of conclusions and future work.

2. **Preliminaries on time scales calculus.**

The origin of calculus on time scales goes back to 1988 [2, 24] when S. Hilger wrote his PhD thesis under the supervision of B. Aulbach. The idea was to unify and extend discrete and continuous analysis. With time this theory proved to be useful in various fields that require modeling of discrete and continuous data simultaneously, like the calculus of variations, control theory, economics and biology (see, e.g. [1, 3, 4, 5, 8, 11, 12, 21, 22, 27, 28, 29]). For a wider look on the time scale calculus and its applications we refer the reader to the books [9, 10].

A time scale, denoted by $\mathbb{T}$, is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. We assume that the time scale $\mathbb{T}$ has the topology inherited from $\mathbb{R}$ with the standard topology. Trivial examples of time scales are $\mathbb{R}$, integer numbers $\mathbb{Z}$ and natural numbers $\mathbb{N}$. Other examples include periodic numbers $h\mathbb{Z} := \{hz : h > 0, z \in \mathbb{Z}\}$; the q-scales $\bar{q^{\mathbb{Z}}} := \{q^k : q > 1, k \in \mathbb{Z}\} \cup \{0\}$ and $q^{\mathbb{N}_0} := \{q^k : q > 1, k \in \mathbb{N}_0\}$; harmonic numbers $\mathbb{T} = \{t_n = \sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\}$; $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{Z}\} \cup \{0\}$; a sequence of disjoint closed intervals.

Let $\sigma : \mathbb{T} \to \mathbb{T}$ be defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M). We call σ the forward jump operator. Analogously we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). Using operators σ and ρ, we can classify points of any time scale $\mathbb{T}$. Namely, a point t is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. We say that t is isolated if $\rho(t) < t < \sigma(t)$, and that t is dense if $\rho(t) = t = \sigma(t)$. The (forward) graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. Hence, for a given $t \in \mathbb{T}$, $\mu(t)$ measures the distance of t to its right neighbor.

Example 2.1. If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{T}$, $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = h\mathbb{Z}$, then for every $t \in \mathbb{Z}$, $\sigma(t) = t + h$, $\rho(t) = t - h$ and $\mu(t) \equiv h$. If $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{Z}\} \cup \{0\}$, then for any $t \in \mathbb{T}$, $\sigma(t) = \frac{t}{1-t}$, $\rho(t) = \frac{t}{1+t}$ and $\mu(t) = \frac{t^2}{1-t}$. If $\mathbb{T} = \{t_n = \sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\}$, then for every $t_n \in \mathbb{T}$, $\sigma(t_n) = t_n + \frac{1}{n+1}$ and $\mu(t_n) = \frac{1}{n+1}$.

In the case when $\sup \mathbb{T}$ is finite and left-scattered, we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{\sup \mathbb{T}\}$. Otherwise, $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The *delta derivative* of f at t is the real number $f^\Delta(t)$ with the property that for any ε there is a neighborhood U of t such that
$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$
for all $s \in U$. We say that f is *delta differentiable* on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Remark 1. If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if f is differentiable in the ordinary sense at t and $f^\Delta(t) = \frac{d}{dt} f(t)$.

If $\mathbb{T}$ is a discrete time scale (all points of $\mathbb{T}$ are isolated), then $f : \mathbb{T} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{T}^\kappa$ with $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.

3. **Main results.** Assume that $\mathbb{T}$ is an isolated time scale such that $\sup\mathbb{T}=\infty$ and $\sup\{\mu(t), t\in\mathbb{T}\}<\infty$. Let us analyze system (2) with initial conditions

$$x(t_0)=x_0=\left(x_1^0,\dots,x_N^0\right)\in\mathbb{E}^N,\quad v(t_0)=v_0=\left(v_1^0,\dots,v_N^0\right)\in\mathbb{E}^N, \tag{4}$$

where $\mathbb{E}=\mathbb{R}^k$ and $t_0\in\mathbb{T}$. System (2) can be rewritten in the matrix form as

$$\begin{cases} x(\sigma(t))=x(t)+\mu(t)v(t)\\ v(\sigma(t))=(I-\mu(t)L)v(t), \end{cases} \tag{5}$$

where I is the $N\times N$ identity matrix, $L=D-A$ is the Laplacian of the matrix $A=\frac{1}{N}(a_{ij})_{i,j=1}^N$ and $D=\frac{1}{N}\mathrm{diag}(d_1,...,d_N)$ with $d_k=\sum_{j=1}^N a_{kj}$. The matrix notation $(I-\mu(t)L)v(t)$ means that the matrix $I-\mu(t)L=(m_{ij})_{i,j=1}^N$ is acting on $\mathbb{E}^N$ by mapping $(v_1,\dots,v_N)$ to $(m_{i1}v_1+\dots+m_{iN}v_N)_{1\le i\le N}$.

Let us define what we mean by a consensus in the context of isolated time scales.

Definition 3.1. We say that system (5) initiated at t_0 tends to a consensus if the consensus parameters tend to a common value, namely $\lim_{n\to\infty}v_i(\sigma^n(t_0))=\bar{v}$, for every $i=1,...,N$.

Remark 2. Note that if there exists $\bar{t}\in\mathbb{T}$ such that $v_i(\bar{t})=\bar{v}$ for every $i=1,...,N$, then for each i the dynamics originating from $(x_i(\bar{t}),\bar{v})$ is given by the rigid translation $x_i(\sigma(t))=x_i(t)+\mu(t)\bar{v}$, meaning that for each $t\ge\bar{t}$ the distance between the states of each two agents is preserved.

In order to establish conditions under which system (5) tends to a consensus we introduce some terminology and recall useful facts.

Proposition 1. [15] *Let $v\in\mathbb{E}^N$. The following are equivalent:*

(i) For $t\in\mathbb{T}$, v is a fixed point of $I-\mu(t)L$, i.e., $(I-\mu(t)L)v=v$;
(ii) $Lv=0$.

Consider

$$\mathcal{Q}_d:=\{(q_1,q_2,...,q_N)\in\mathbb{E}^N : q_1=q_2=\dots=q_N\},$$

the diagonal of $\mathbb{E}^N$ and

$$\mathcal{Q}_d^\perp:=\{(q_1,q_2,...,q_N)\in\mathbb{E}^N : q_1+q_2+\dots+q_N=0\},$$

its orthogonal complement with respect to the standard inner product. Therefore, every $v\in\mathbb{E}^N$ can be uniquely written as $v=v_d+v_\perp$, with $v_d\in\mathcal{Q}_d$ and $v_\perp\in\mathcal{Q}_d^\perp$.

It is immediate to check that $L(\mathcal{Q}_d)=0$ and $L(\mathcal{Q}_d^\perp)\subseteq\mathcal{Q}_d^\perp$. It means that, if $v(\sigma(t))=(I-\mu(t)L)v(t)$, then $v_\perp(\sigma(t))=(I-\mu(t)L)v_\perp(t)$. The same holds for the state parameter. This implies that the projection onto $\mathcal{Q}_d^\perp$ of the solutions of (5) are the solutions of the restriction of the system to $\mathcal{Q}_d^\perp$.

Observe that convergence of all v_i to a common value (see Definition 3.1) means convergence to the diagonal of $\mathbb{E}^N$ or, if we set $Q=\mathbb{E}^N/\mathcal{Q}_d$, convergence to 0 in this quotient space.

Proposition 2. [13] *The following definitions of consensus are equivalent:*

(i) $\lim_{n\to\infty}v_i(\sigma^n(t_0))=\bar{v}$, for $i=1,...,N$;
(ii) $\lim_{n\to\infty}v_{\perp i}(\sigma^n(t_0))=0$, for $i=1,...,N$.

In $Q \simeq \mathcal{Q}_d^{\perp}$, fix an inner product

$$\langle p, q\rangle_Q = \frac{1}{2}\sum_{i,j=1}^{N}\langle p_i - p_j, q_i - q_j\rangle$$

and denote by $\|\cdot\|_Q$ the norm induced by this inner product.

Now we are in position to formulate our main result giving a sufficient condition under which system (5) converges to a consensus.

Theorem 3.2. *If there exists $0 \le c < 1$ such that $\big|\big|I - \mu(t)L\big|\big|_Q \le c$ for all $t \ge t_0$, then $\big|\big|v(\sigma^n(t_0))\big|\big|_Q \to 0$, when $n \to \infty$. Moreover, there exists a constant C such that $\big|\big|x(\sigma^n(t_0))\big|\big|_Q \le C$, for all $n \in \mathbb{N}$, and there exists $\hat{x} \in \mathcal{Q}_d^{\perp}$ such that $x_\perp(\sigma^n(t_0)) \to \hat{x}$, when $n \to \infty$.*

Proof. For all $n \in \mathbb{N}$ we have the following evaluation:

$$\begin{aligned}||v(\sigma^n(t_0))||_Q &= ||(I - \mu(\sigma^{n-1}(t_0))L)v(\sigma^{n-1}(t_0))||_Q\\ &\le ||I - \mu(\sigma^{n-1}(t_0))L||_Q \cdot ||v(\sigma^{n-1}(t_0))||_Q\\ &\le c||v(\sigma^{n-1}(t_0))||_Q \le c^n||v(t_0)||_Q.\end{aligned}$$

Therefore,

$$||v(\sigma^n(t_0))||_Q \le c^n||v(t_0)||_Q \ \to 0 \ \text{ for } \ n \to \infty\,,$$

what finishes the first part of the proof.

Now let us evaluate the norm of $x(\sigma^n(t_0))$:

$$\begin{aligned}\|x(\sigma^n(t_0))\|_Q &= ||x(t_0) + \sum_{i=0}^{n-1}\mu(\sigma^i(t_0))v(\sigma^i(t_0))||_Q\\ &\le \|x(t_0)\|_Q + s\sum_{i=0}^{n-1}c^i\|v(t_0)\|_Q = \|x(t_0)\|_Q + s\|v(t_0)\|_Q\cdot\frac{1-c^n}{1-c},\end{aligned}$$

where $s = \sup\{\mu(t), t \in \mathbb{T}\}$. Since $c \in \big[0,1\big)$, we have

$$\|x(\sigma^n(t_0))\|_Q \le \|x(t_0)\|_Q + \frac{s}{1-c}\|v(t_0)\|_Q,$$

proving that $\|x(\sigma^n(t_0))\|_Q$ is bounded for all $n \in \mathbb{N}$. Finally, for $m > n$,

$$\begin{aligned}\|x(\sigma^m(t_0)) - x(\sigma^n(t_0))\|_Q &= \Big\|\sum_{i=0}^{m-1}\mu(\sigma^i(t_0))v(\sigma^i(t_0)) - \sum_{i=0}^{n-1}\mu(\sigma^i(t_0))v(\sigma^i(t_0))\Big\|_Q\\ &= \Big\|\sum_{i=n}^{m-1}\mu(\sigma^i(t_0))v(\sigma^i(t_0))\Big\|_Q \le s\sum_{i=n}^{m-1}c^i\|v(t_0)\|_Q\\ &= s\frac{c^n - c^m}{1-c}\|v(t_0)\|_Q \le s\frac{c^n}{1-c}\|v(t_0)\|_Q\,.\end{aligned}$$

Therefore, $\big(x\big(\sigma^n(t_0)\big)\big)_{n\in\mathbb{N}}$, as a Cauchy sequence, is convergent. ∎

Example 3.3. Let us consider system (5) with $N = 5$, $\mathbb{E} = \mathbb{R}$, and the matrix $A = \frac{1}{5}\bar{A}$, where $\bar{A}$ is given by (3). In this case the Laplacian matrix

$$L = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

has the following eigenvalues: $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{5}$, $\lambda_5 = \frac{2}{5}$ with the corresponding eigenvectors: $w_1 = (1,1,1,1,1)$, $w_2 = (0,1,0,0,1)$, $w_3 = (0,0,1,0,0)$, $w_4 = (0,0,0,1,0)$, $w_5 = (0,0,0,0,1)$. Therefore,

$$Q = \mathbb{R}^5/\mathcal{Q}_d \simeq span\{w_2, w_3, w_4, w_5\}.$$

Moreover, $\|I - \mu L\|_Q < 1$ if and only if $\mu \in (0,5)$. This explains why on the time scale $\mathbb{T} = \left\{1 - \frac{1}{k} \,:\, k = 1,\dots,10\right\} \cup \left\{t_k = 1 + 2.5\sum_{i=0}^{k}|\sin(i)| \,:\, k \in \mathbb{N}_0\right\}$, where $\mu(t) < 5$ for all $t \in \mathbb{T}$, the system tends to a consensus (see Figure 1), while on $\mathbb{T} = \left\{1 - \frac{1}{k} \,:\, k = 1,\dots,10\right\} \cup \left\{t_k = 1 + 6\sum_{i=0}^{k}|\sin(i)| \,:\, k \in \mathbb{N}_0\right\}$ a consensus is not ensured (see Figure 2).

Generally, checking the condition $\|I - \mu(t)L\|_Q < 1$, for all $t \geq t_0$, is computationally cumbersome. However, there are two special cases that are easier to analyze. Namely, when A is a symmetric matrix or A is such that $\sum_{j=1}^{N} a_{ij} = N$, for all $i = 1,\dots,N$, with all its rows equal.

Recall that the matrix norm induced by the Euclidean norm coincides with the spectral norm. That is, if A is a $N \times N$ real matrix, then

$$\|A\| = \sqrt{\sigma_{max}(A)},$$

where $\sigma_{max}(A)$ is the largest eigenvalue of the symmetric matrix A^TA. In particular, if A is symmetric, then the Laplacian L is also symmetric and its eigenvalues $l_1,\dots,l_N$ satisfy

$$0 = l_1 \leq l_2 \leq \cdots \leq l_N = \|L\|.$$

In this case

$$\|I - \mu(t)L\|_Q = \max_{\lambda_i(\mu(t))\neq 1}\left\{|\lambda_1(\mu(t))|,\dots,|\lambda_m(\mu(t))|\right\} = |\lambda(\mu(t))|,$$

where $\left\{\lambda_1(\mu(t)),\dots,\lambda_m(\mu(t))\right\}$ denotes the set of all distinct eigenvalues of $I - \mu(t)L$.

Corollary 1. *If A is a symmetric matrix and $|\lambda(\mu(t))| < 1$, for all $t \geq t_0$, then system (2) with the initial conditions (4) tends to the consensus $(\bar{v},\bar{v},\dots,\bar{v})$, where $\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i^0$.*

Proof. Observe that the average mean of the consensus parameters is invariant in time. Indeed, summing from $i = 1$ to N both sides of equations

$$v_i(\sigma(t)) = v_i(t) + \frac{\mu(t)}{N}\sum_{j=1}^{N} a_{ij}\left(v_j(t) - v_i(t)\right)$$

and using symmetry of matrix A, we get

$$\sum_{i=1}^{N} v_i(\sigma(t)) = \sum_{i=1}^{N} v_i(t).$$

It means that

$$\sum_{i=1}^{N} v_i(\sigma^n(t_0)) = \sum_{i=1}^{N} v_i^0,$$

for all $n \in \mathbb{N}$. Taking the limit $(n \to \infty)$ on both sides of the last equation one obtains $\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i^0$. $\square$

Example 3.4. Let us consider system (5) with $N = 5$, $\mathbb{E} = \mathbb{R}$ and the symmetric matrix

$$A = \frac{1}{5}\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 & 3 \end{bmatrix}.$$

Using a software, it can be checked that the interval $(0, 1.2871)$ is an approximate solution to $\|I - \mu L\|_Q < 1$. Figures 3–4 illustrate the time evolution of consensus parameters and their states on different time scales when $0 < \mu(t) < 1.2871$, for all $t \in \mathbb{T}$. In all those cases, the system tends to a consensus. Figure 5 shows the situation for $\mu = 1.2871$, where there is no consensus. However, when we choose the time scale with $\mu = 1.2771, 1.2871, 1.2771, 1.2871, \ldots$, simulations show that the system tends to a consensus (Figure 6).

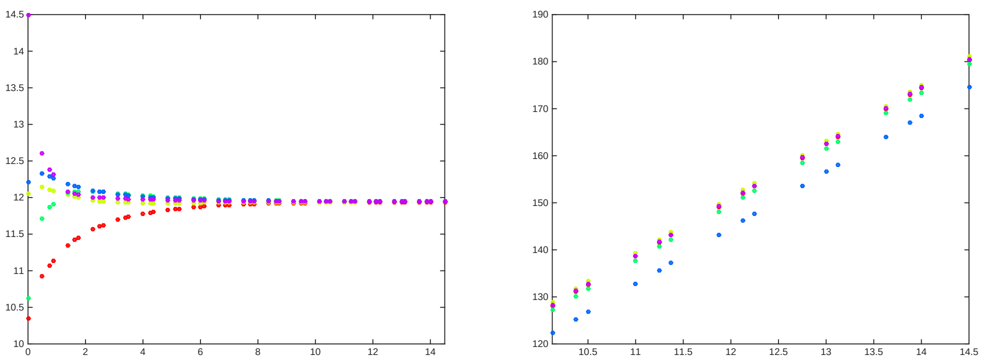

FIGURE 3. Time evolution of 5 consensus parameters with 50 iterations (left) and their states in the last 16 iterations (right) on $\mathbb{T} = \{0; 0.5; 0.75; 0.875; 1.375; 1.625; \ldots\}$.

Now let us consider the second special case when matrix A is such that $\sum_{j=1}^{N} a_{ij} = N$, for all $i = 1, \ldots, N$, and all the rows of A are equal.

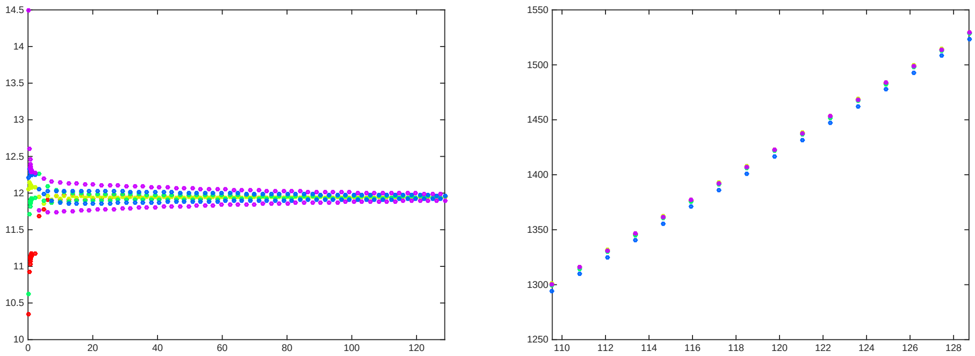

FIGURE 4. Time evolution of 5 consensus parameters with 150 iterations (left) and their states in the last 16 iterations (right) on $\mathbb{T} = \left\{1 - \frac{1}{k} \;:\; k = 1, \dots, 50\right\} \cup \{1 + 1.2771k \;:\; k \in \mathbb{N}_0\}$.

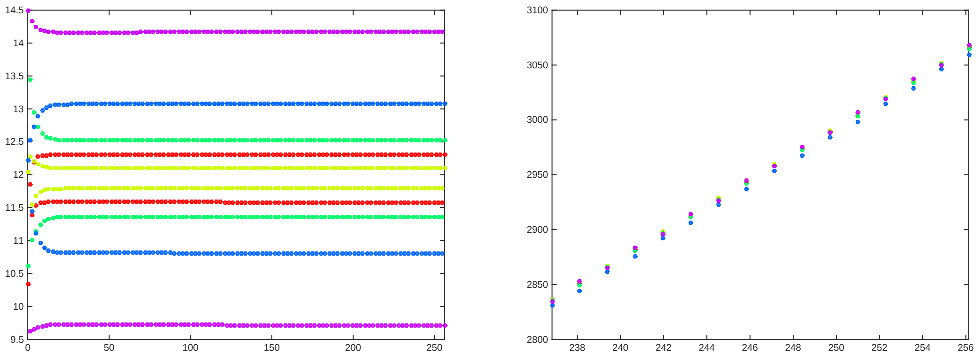

FIGURE 5. Time evolution of 5 consensus parameters with 200 iterations (left) and their states in the last 16 iterations (right) on $\mathbb{T} = 1.2871\mathbb{N}_0$.

Proposition 3. *Assume that the matrix A satisfies $\sum_{j=1}^{N} a_{ij} = N$, for all $i = 1, \dots, N$ and all its rows are equal. If $0 < \mu(t) < 2$, for all $t \geq t_0$, then system* (5) *with the initial conditions* (4) *converges to the consensus $(\bar{v}, \bar{v}, \dots, \bar{v})^T = Av_0^T$.*

Proof. By assumptions, the Laplacian of the matrix A can be written as

$$L = \begin{bmatrix} 1 - \frac{a_1}{N} & -\frac{a_2}{N} & \dots & -1 + \frac{\sum_{j=1}^{N-1} a_j}{N} \\ -\frac{a_1}{N} & 1 - \frac{a_2}{N} & \dots & -1 + \frac{\sum_{j=1}^{N-1} a_j}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_1}{N} & -\frac{a_2}{N} & \dots & \frac{\sum_{j=1}^{N-1} a_j}{N} \end{bmatrix},$$

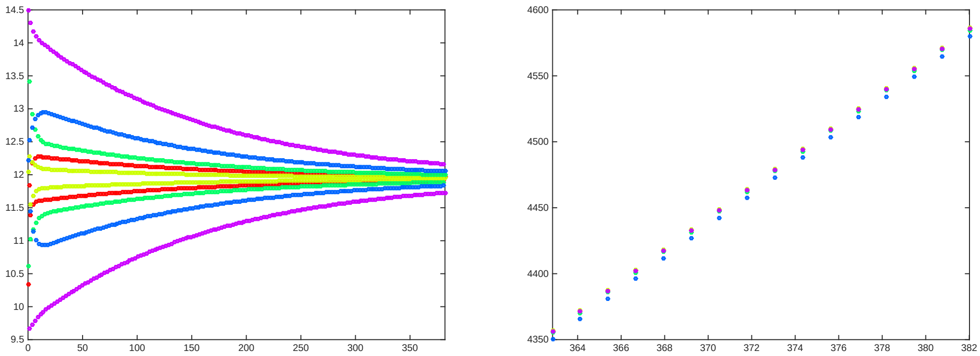

FIGURE 6. Time evolution of 5 consensus parameters with 300 iterations (left) and their states in the last 16 iterations (right) on the time scale when $\mu = 1.2771, 1.2871, 1.2771, 1.2871, \ldots$.

where, for convenience, $a_j = a_{ij}$ for all $i, j = 1, \ldots, N$. Therefore,

$$\det\big(I - \mu(t)L - \lambda(t)I\big) = \big(1 - \lambda(t)\big)\big(1 - \mu(t) - \lambda(t)\big)^{N-1},$$

meaning that

$$\big\|I - \mu(t)L\big\|_Q = |1 - \mu(t)|.$$

Since $0 < \mu(t) < 2$, for all $t \geq t_0$, we have $|1 - \mu(t)| < 1$, for all $t \geq t_0$. Finally, applying Theorem 3.2 we conclude that system (5) tends to a consensus. By the same method as in the proof of Corollary 1 one gets the following equality

$$\sum_{i=1}^{N} a_i v_i(\sigma^n(t_0)) = \sum_{i=1}^{N} a_i v_i^0,$$

for all $n \in \mathbb{N}$. Now to finish the proof it is enough to take the limit $(n \to \infty)$ on both sides of the last equation. $\square$

Example 3.5. In order to illustrate Proposition 3 let us consider system (5) with $N = 30$, $\mathbb{E} = \mathbb{R}$ and a randomly chosen matrix A. On $\mathbb{T} = \left\{t_n = \sum_{k=1}^{n} \frac{1}{k} \ : \ n \in \mathbb{N}\right\}$ with $\mu(t_n) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$, the system tends to a consensus (Figure 7). Figure 8 shows the behavior of the system on the time scale with $\mu = \frac{1}{4}, \frac{5}{2}, 2, \frac{1}{4}, \frac{5}{2}, 2, \ldots$, where there is no consensus. However, on the time scale with $\mu = \frac{1}{4}, \frac{3}{4}, 2, \frac{1}{4}, \frac{3}{4}, 2, \ldots$, simulations show that the system tends to a consensus (Figure 9).

Remark 3. Observe that numerical simulations in Example 3.4 and Example 3.5 demonstrate that the condition given in Theorem 3.2 is not necessary for system (5) to tend to a consensus (Figure 6 and Figure 9).

Remark 4. In the special case when $\mathbb{E} = \mathbb{R}$, $a_{ij} = 1$ and $|v_j(t) - v_i(t)| < 1$, for all $i, j = 1, \ldots, N$, the equation for the consensus parameters in model (2) coincides with the Krause model on isolated time scales (see [20]):

$$v_i^{\Delta}(t) = \frac{1}{\sum_{j:|v_j(t)-v_i(t)|<1} 1} \sum_{j:|v_j(t)-v_i(t)|<1} \left(v_j(t) - v_i(t)\right). \tag{6}$$

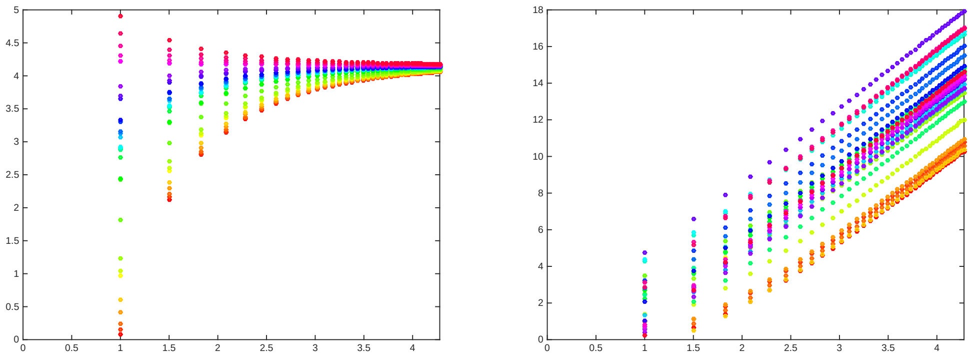

Figure 7. Time evolution of 30 consensus parameters with 40 iterations (left) and their states (right) on $\mathbb{T} = \left\{t_n = \sum_{k=1}^{n} \frac{1}{k} \ : \ n \in \mathbb{N}\right\}$.

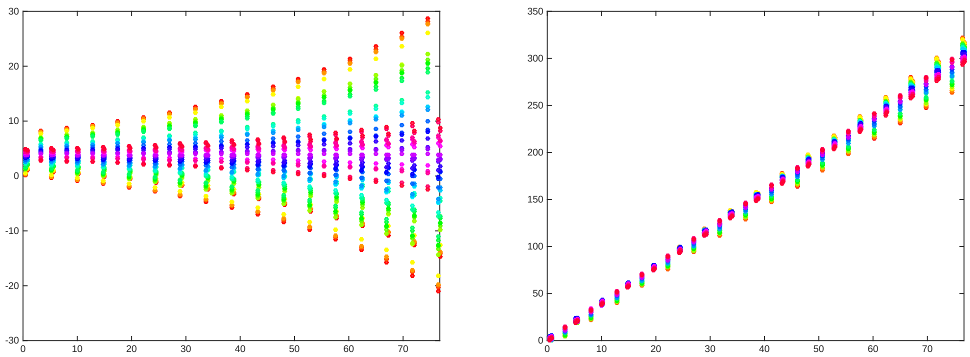

Figure 8. Time evolution of 30 consensus parameters with 50 iterations (left) and their states (right) on the time scale when $t_0 = 0$ and $\mu = \frac{1}{4}, \frac{5}{2}, 2, \frac{1}{4}, \frac{5}{2}, 2, \ldots$.

Observe that when $|v_j(t) - v_i(t)| < 1$, for all $i, j = 1, \ldots, N$ and $\mu(t) \in \big(0, 2\big)$, for all $t \geq t_0$, then by Proposition 3 system (6) initiated at t_0 tends to a consensus. This improves the sufficient condition for the Krause model to reach a consensus presented in [20] for the case when all agents belong to the same confidence set (i.e., everyone influences on all the others opinions).

4. **Conclusions.** It is well known that the continuous and the discrete time models behave sometimes differently, sometimes similarly. However, even in the discrete case when using different time measurements the dynamic behavior of a model can be changed. Here one may consider as an example the Cucker–Smale type model proposed in this paper. The theory of time scales proved to be a powerful mathematical tool that allows to analyze the dynamic behavior of a model as the time measurement is changed. Therefore it was natural to apply this theory to the Cucker–Smale model. We have proved a sufficient condition under which the

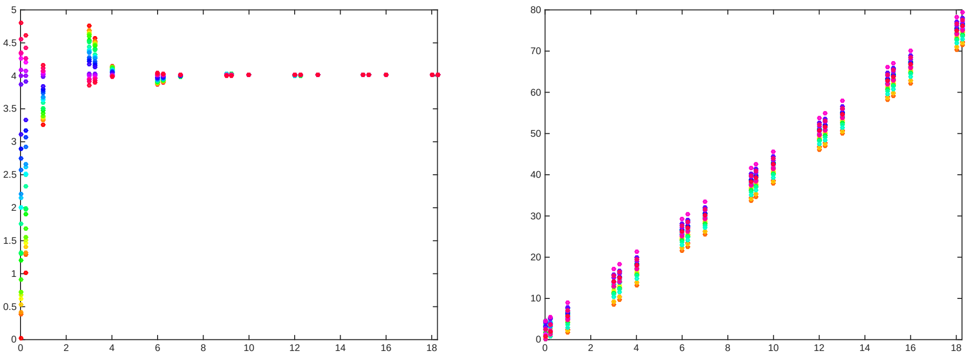

FIGURE 9. Time evolution of 30 consensus parameters with 20 iterations (left) and their states (right) on the time scale when $t_0 = 0$ and $\mu = \frac{1}{4}, \frac{3}{4}, 2, \frac{1}{4}, \frac{3}{4}, 2, \ldots$.

Cucker–Smale type model tends to a consensus. This condition shows clearly the importance of the graininess function in a consensus phenomenon. The advantage of using this result lies in the fact that in applications the graininess function can be interpreted as a frequency of meetings of the group members. We want to emphasize that this work was intended as an attempt to motivate studies on the Cucker–Smale model by using the theory of time scales. There are many open problems: How to compute the maximum value of μ, for an arbitrary matrix A, in order to satisfy the hypothesis of Theorem 3.2? How to establish a sufficient condition for having a consensus in the case when $\mathbb{T}$ is an arbitrary time scale? How to prove a counterpart of Theorem 3.2 in the case when $\mathbb{T}$ is an arbitrary time scale and the weights a_{ij} are functions of x_i, x_j? Those are directions of our further research.

Acknowledgments. E. Girejko and A. B. Malinowska were supported by the Polish National Science Center grant on the basis of decision DEC-2014/15/B/ST7/05270. N. Martins was supported by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (CIDMA), and *The Portuguese Foundation for Science and Technology* (FCT), within project UID/MAT/04106/2013.

REFERENCES

[1] F. M. Atici, D. C. Biles and A. Lebedinsky, An application of time scales to economics, *Math. Comput. Modelling*, **43** (2006), 718–726.

[2] B. Aulbach and S. Hilger, A unified approach to continuous and discrete dynamics, in *Qualitative theory of differential equations (Szeged, 1988)*, 37–56, Colloq. Math. Soc. J. Bolyai, 53, North-Holland, Amsterdam, 1990.

[3] Z. Bartosiewicz, N. Martins and D. F. M. Torres, The second Euler–Lagrange equation of variational calculus on time scales, *European Journal of Control*, **17** (2011), 9–18.

[4] Z. Bartosiewicz, Linear positive control systems on time scales: controllability, *Mathematics of Control, Signals and Systems*, **25** (2013), 327–343.

[5] J. Belikov, U. Kotta and M. Tonso, Realization of nonlinear MIMO system on homogeneous time scales, *European Journal of Control*, **23** (2015), 48–54.

[6] V. D. Blondel, J. M. Hendrickx and J. N. Tsitsiklis, On Krause's multi-agent consensus model with state-dependent connectivity, *IEEE Transactions on automatics control*, **54** (2009), 2586–2597.

[7] V. D. Blondel, J. M. Hendrickx and J. N. Tsitsiklis, Continuous-time average-preserving opinion dynamics with opinion-dependent communications, *SIAM J. Control Optim.*, **18** (2010), 5214–5240.

[8] M. Bohner, M. Fan and J. Zhang, Periodicity of scalar dynamic equations on time scales and applications to population models, *J. Math. Anal. Appl.*, **330** (2007), 1–9.

[9] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001.

[10] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003.

[11] M. Bohner and H. Warth, The Beverton-Holt dynamic equation, *Appl. Anal.* **86** (2007), 1007–1015.

[12] D. Casagrande, U. Kotta, M. Tonso and M. Wyrwas, Transfer equivalence and realization of nonlinear input-output delta-differential equations on homogeneous time scales, *IEEE Transactions on Automatic Control*, **55** (2010), 2601–2606.

[13] M. Caponigro, M. Fornasier, B. Piccoli and E. Trélat, Sparse stabilization and optimal control of the Cucker-Smale model, *Mathematics of Control and Related Fields*, **3** (2013), 447–466.

[14] S. Chatterjee and E. Seneta, Towards Consensus: Some Convergence Theorems on Repeated Averaging, *J. Appl. Prob.*, **14** (1977), 159–164, 1977.

[15] F. Cucker and S. Smale, Emergent Behavior in Flocks, *IEEE Transactions on Automatic Control*, **52** (2007), 852–862.

[16] F. Cucker and S. Smale, On the mathematics of emergence, *Japanese Journal of Mathematics*, **2** (2007), 197–227.

[17] M. H. DeGroot, Reaching a Consensus. *Journal of the American Statistical Association*, **69** (1974), 118–121.

[18] J. French, A formal theory of social power, *Psychological Review*, **63** (1956), 181–194.

[19] G. Fu, W. Zhang and Z. Li, Opinion dynamics of modified Hegselmann–Krause model in a group-based population with heterogeneous bounded confidence,*Physica A*, **419** (2015), 558–565.

[20] E. Girejko, L. Machado, A. B. Malinowska and N. Martins, Krause's model of opinion dynamics on isolated time scales, *Mathematical Methods in the Applied Sciences*, DOI: 10.1002/mma.3916.

[21] E. Girejko, A. B. Malinowska and D. F. M. Torres, The contingent epiderivative and the calculus of variations on time scales, *Optimization Letters*, **61** (2012), 251–264.

[22] E. Girejko and D. F. M. Torres, The existence of solutions for dynamic inclusions on time scales via duality, *Applied Mathematic Letters*, **25** (2012), 1632–1637.

[23] R. Hegselmann and U. Krause, Opinion dynamics and bounded confidence: models, analysis, and simulation, *J. Artificial Societies and Social Simulations*, **5** (2002), 1–33.

[24] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, Ph.D thesis, Universität Würzburg, 1988.

[25] U. Krause, A discrete nonlinear and non-autonomous model of consensus formation, *Comunications in Difference Equations* (Eds. S. Elaydi, G. Ladas, J. Popenda, J. Rakowski), Gordon and Breach Publ., Amsterdam, 2000, 227–236.

[26] K. Lehrer and C. Wagner, *Rational Consensus in Science and Society*, D. Reidel Publishing Company, Dordrecht, Holland, 1981.

[27] A. B. Malinowska and D. F. M. Torres, Natural boundary conditions in the calculus of variations, *Math. Methods Appl. Sci.*, **33** (2010), 1712–1722.

[28] A. B. Malinowska, N. Martins and D. F. M. Torres, Transversality conditions for infinite horizon variational problems on time scales, *Optim. Lett.*, **5** (2011), 41–53.

[29] N. Martins and D. F. M. Torres, Calculus of variations on time scales with nabla derivatives, *Nonlinear Anal.*, **71** (2009), e763–e773.

[30] H. Wang and L. Shang, Opinion dynamics in networks with common-neighbors-based connections, *Physica A* , **421** (2015), 180–186.

[31] T. Vicsek, A. Czirók, E. Ben-Jacob and O. Shochet, Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Letters*, **75** (1995), 1226–1229.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: e.girejko@pb.edu.pl
E-mail address: lmiguel@utad.pt
E-mail address: a.malinowska@pb.edu.pl
E-mail address: natalia@ua.pt